Existence and uniqueness of solutions for a nonlocal parabolic thermistor-type problem*

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Abstract

In this paper we prove existence and uniqueness of solutions to the nonlocal parabolic problem

$$\frac{\partial u}{\partial t} - \triangle_p u = \lambda \frac{f(u)}{\left(\int_\Omega f(u)\,dx\right)^2}, \quad \text{in } \Omega\times]0,T[\,,$$

which generalizes the electric heating problem of a conducting body.

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1 Introduction

In this paper we study the existence and uniqueness of bounded solutions for the following nonlocal parabolic problem:

$$\begin{aligned} \frac{\partial u}{\partial t} - \triangle_p u = \lambda \frac{f(u)}{\left(\int_\Omega f(u)\,dx\right)^2}\,, &\quad \text{in } \Omega\times]0,T[\,, \\ u = 0 \quad \text{on } \partial\Omega\times]0,T[\,, & \\ u(0) = u_0 \quad \text{in } \Omega\,, & \end{aligned} \tag{1}$$

where $\Delta_p = div(-|\nabla u|^{p-2}\nabla u)$; $p \geq 2$; $T > 0$; $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a regular bounded domain; λ a positive parameter; and f a function from $\mathbb{R}$ to $\mathbb{R}$ with prescribed conditions.

For $p = 2$, Δ_p is reduced to the usual Laplacian operator, and problem (1) serves as a model for the well-known and important thermistor problem, where u is the temperature inside a conductor – see e.g. (Lacey, 1995*a*; Lacey, 1995*b*; Bebernes and Lacey, 1997; Tzanetis, 2002). This problem is very important in industry and engineering applications, and has attracted attention in the literature over the last decade, from both the experimental and theoretical point of views: see (Antontsev and Chipot, 1994; Allegretto *et al.*, 1999; El Hachimi and Sidi Ammi, 2002; González Montesinos and Ortegón Gallego, 2002; El Hachimi and Sidi Ammi, 2005; Kutluay and Esen, 2005) and references therein.

Our main result is a proof of the global existence and uniqueness of solutions of problem (1). The result is a generalization of (Lacey, 1995*a*; Lacey, 1995*b*; Tzanetis, 2002; El Hachimi and Sidi Ammi, 2005) to the general p-Laplacian case, $p \geq 2$. For the particular case $p = 2$, the result is obtained in (El Hachimi and Sidi Ammi, 2005), but under somehow less restrictive assumptions on the data of the problem: Theorem 2 does not impose restrictions on α, while in (El Hachimi and Sidi Ammi, 2005) it is assumed that $\alpha < \frac{4}{N-2}$, $N > 2$.

2 Existence and uniqueness

The definition of solution for problem (1) is understood in the standard way.

Definition 1 *We say that u is a solution of (1) if, and only if,*

$$u \in L^\infty(\tau, +\infty, W_0^{1,p}(\Omega) \cap L^\infty(\Omega))$$

with $\frac{\partial u}{\partial t} \in L^2(\tau, +\infty, L^{p'}(\Omega))$ for any $\tau > 0$, and the following equation is satisfied for all $\phi \in C^\infty((0,\infty), \Omega)$:

$$\int_0^T \int_\Omega u \frac{\partial}{\partial t}\phi - |\nabla u|^{p-2}\nabla u \nabla\phi \,dxdt = \int_0^T \left(\frac{\lambda}{\left(\int_\Omega f(u)\,dx\right)^2} \int_\Omega f(u)\phi dx \right) dt\,.$$

The main result of the paper is as follows.

Theorem 2 *Let the hypotheses (H1) and (H2) be satisfied:*

(H1) $f : \mathbb{R} \to \mathbb{R}$ *is a locally Lipschitzian function;*

(H2) There exist positive constants c_1, c_2 *and* α *such that for all* $\xi \in \mathbb{R}$

$$\sigma \le f(\xi) \le c_1|\xi|^{\alpha+1} + c_2 \,.$$

Further, assume that $u_0 \in L^{k_0+2}(\Omega)$ *with*

$$k_0 \ge \max\left(0, \frac{N(\alpha+2-p)}{p} - 2\right) . \tag{2}$$

Then, there exists a constant $d_0 > 0$ *such that if* $\|u_0\|_{k_0+2} < d_0$*, the problem (1) admits a solution* u *verifying*

$$u \in L^\infty(\tau, +\infty, L^{k_0+2}(\Omega)) \,,$$
$$|u|^\gamma u \in L^\infty(\tau, +\infty, W_0^{1,p}(\Omega)), \ \textit{with } \gamma = \frac{k_0}{p} \,,$$

for all $\tau > 0$*. Moreover, if* $u_0 \in L^\infty(\Omega)$*, then* $u \in L^\infty(\tau, +\infty, L^\infty(\Omega))$ *and* u *is unique.*

Remark 3 *A value for* d_0 *is given explicitly in the proof of Theorem 2 – cf.* (8).

3 Proof of Theorem 2

The existence is proved by the Faedo-Galerkin method.

3.1 Existence

Let $w_1, \ldots, w_m, \ldots$ be a complete sequence of linearly independent elements of $H_0^1(\Omega)$. For each m, we define an approximate solution

$$u_m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j$$

of (1), where g_{jm} are solutions of the following system of ordinary differential equations:

$$\langle u_m', w_j\rangle + (u_m, w_j) = \frac{\lambda}{\left(\int_\Omega f(u_m)\, dx\right)^2} \langle f(u_m), w_j\rangle \,, \tag{3}$$

$j = 1, \ldots, m$, with the initial condition

$$u_m(0) = u_{om} \,, \tag{4}$$

u_{om} being the orthogonal projection in $H_0^1(\Omega)$ of u_0 on the space spanned by $w_1, \ldots, w_m$. The initial-value problem (3)-(4) is equivalent to a linear m-dimensional ordinary differential equation for the g_{jm}. The existence and uniqueness of the g_{jm} on a maximal interval $[0, t_m[$ is obvious. We obtain the existence of a solution u for our problem (1) passing to the limit, as $m \to \infty$. For that we need to derive a priori estimates on u_m which guarantee that $t_m = T$. This is done by Lemma 5. In order to prove it, we employ an inequality due to Ghidaglia.

Lemma 4 (Ghidaglia inequality) *Let y be a positive absolutely continuous function on $(0, +\infty)$ which satisfies*

$$y' + \gamma y^\nu \leq \delta \,,$$

with $\nu > 1, \gamma > 0$ and $\delta \geq 0$. Then,

$$y(t) \leq \left(\frac{\delta}{\gamma}\right)^{\frac{1}{\nu}} + (\gamma(\nu-1)t)^{-\frac{1}{(\nu-1)}} \,,$$

for all $t \geq 0$,

Proof of Lemma 4 can be found in (Temam, 1997).

Lemma 5 *For any $\tau > 0$, there exist constants $c_3(\tau)$ and $c_4(\tau)$ such that for all $t \geq \tau$*

$$\|u_m(t)\|_{k_0+2} \leq c_3(\tau) \,, \tag{5}$$

$$\|u_m(t)\|_\infty \leq c_4(\tau) \,. \tag{6}$$

Remark 6 *Throughout the paper we denote by c_i different positive constants, which depend on the data of the problem, but not on m.*

Proof. Multiplying the equation (3) by $|u_m|^k g_{jm}$, integrating on Ω, summing up for $j = 1, \ldots, m$ and using (H1)-(H2), yields

$$\frac{1}{k+2}\frac{d}{dt}\|u_m\|_{k+2}^{k+2} + \frac{p^p}{(k+p)^p}\|\nabla(|u_m|^{\frac{k}{p}}u_m)\|_p^p \leq c_5\|u_m\|_{k+\alpha+2}^{k+\alpha+2} + c_6 \,. \tag{7}$$

By using condition (2) on k_0 and well-known Sobolev's and Gagliardo-Nirenberg's inequalities, we obtain

$$\left(c_7\|u_m\|_{k_0+2}^\alpha - \frac{4}{(k_0+p)^p}\right)\|\nabla|u_m|^\gamma u_m\|_p^p + c_6 \geq \frac{1}{k_0+2}\frac{d}{dt}\|u_m\|_{k_0+2}^{k_0+2} \,.$$

Using the compatibility condition on u_0

$$\|u_0\|_{k_0+2} < \left(\frac{4}{c_7(k_0+p)^p}\right)^{\frac{1}{\alpha}} = d_0 \,, \tag{8}$$

and the continuity of u_m, there exists a small $\tau > 0$ such that

$$\frac{1}{k_0+2}\frac{d}{dt}\|u_m\|_{k_0+2}^{k_0+2} + c_8\|\nabla(|u_m|^\gamma u_m)\|_p^p \le c_6 \tag{9}$$

for all $0 < t < \tau$. Setting

$$y_{k_0}(t) = \|u_m\|_{k_0+2}^{k_0+2}$$

and using the Poincaré and Holder inequalities on the left side of (9), there exist two constants $\gamma > 0$ and $\delta > 0$ such that

$$\frac{dy_{k_0}}{dt} + \gamma y_{k_0}^{\frac{k_0+p}{k_0+2}} \le \delta$$

for all $0 < t < \tau$. Note that for $p > 2$ we have $\frac{k_0+p}{k_0+2} > 1$. Estimate (5) follows from Lemma 4.

The proof of (6) is similar to the proof of inequality (2.4) in (El Hachimi and Sidi Ammi, 2005), and is given here for completeness. By using Holder's inequality, we get

$$\|u_m\|_{k+\alpha+2}^{k+\alpha+2} \le c_9\|u_m\|_{k+2}^{\theta_1}\|u_m\|_{k_0+2}^{\theta_2}\|u_m\|_q^{\theta_3}\,, \tag{10}$$

with θ_1, θ_2 and θ_3 satisfying

$$\frac{\theta_1}{k+2} + \frac{\theta_2}{k_0+2} + \frac{\theta_3}{q} = 1$$

and

$$\theta_1 + \theta_2 + \theta_3 = k + \alpha + 2.$$

Moreover, we require

$$\frac{\theta_1}{k+2} + \frac{\theta_3}{p(\gamma+1)} = 1.$$

Using the boundedness of $\|u_m\|_{k_0+2}$, the choice of q, Sobolev and Young's inequalities and relation (10), we derive

$$\begin{aligned} c_5\|u_m\|_{k+\alpha+2}^{k+\alpha+2} &\le c_{10}\|u_m\|_{k+2}^{\theta_1}\|\nabla|u_m|^\gamma u_m\|_p^{\frac{\theta_3}{\gamma+1}} \\ &\le c_{11}(k+2)^{\theta_4}\|u_m\|_{k+2}^{k+2} + \frac{p^p}{2(k+p)^p}\|\nabla|u_m|^\gamma u_m\|_2^2\,, \end{aligned}$$

where θ_4 is some positive constant. Hence (7) becomes

$$\frac{1}{k+2}\frac{d}{dt}\|u_m\|_{k+2}^{k+2} + \frac{c_{12}}{(k+p)^p}\|\nabla|u_m|^\gamma u_m\|_p^p \le c_{13}(k+p)^{\theta_4}\|u_m\|_{k+2}^{k+2} + c_6.$$

Therefore, by applying Lemma 4 of (Filo, 1990), we conclude (6). ■

Multiplying the jth equation of system (3) by $g_{jm}(t)$, summing these equations for $j = 1, \ldots, m$ and integrating with respect to the time variable, we deduce the existence of a subsequence of u_m such that

$$\begin{aligned} u_m &\to u \text{ weak star in } L^\infty(0,T;L^2(\Omega))\,, \\ u_m &\to u \text{ weak in } L^2(0,T;W_0^{1,p}(\Omega))\,, \\ u_{mt} &\to u_t \text{ weak in } L^2(0,T;W^{-1,p'}(\Omega))\,, \\ u_m &\to u \text{ strongly in } L^p(0,T;L^p(\Omega))\,. \end{aligned}$$

Standard compactness and monotonicity arguments allow us to assert that u is a solution of problem (1).

3.2 Uniqueness

Let u_1 and u_2 be two weak solutions of problem (1), and define $w = u_1 - u_2$. Subtracting the equations verified by u_1 and u_2, we obtain:

$$\begin{aligned} \frac{dw}{dt} - (\triangle_p u_2 - \triangle_p u_1) = {} & \frac{\lambda\left(f(u_1) - f(u_2)\right)}{\left(\int_\Omega f(u_1)\,dx\right)^2} \\ & + \lambda \frac{\left(\int_\Omega f(u_2) - f(u_1)\,dx\right)\left(\int_\Omega f(u_2) + f(u_1)\,dx\right)}{\left(\int_\Omega f(u_1)\,dx\right)^2 \left(\int_\Omega f(u_2)\,dx\right)^2} f(u_2)\,. \end{aligned}$$

Taking the inner product of last equation by w and using (H1), (6), and the monotonicity of the p-Laplacian, we get

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 \le c_{14}\|w(t)\|_2^2\,,$$

which implies that $w = 0$. Hence, the solution is unique.

4 Absorbing sets and attractors

We denote by $\{S(t), t \ge 0\}$ the continuous semi-group generated by (1) and defined by

$$\begin{aligned} S(t) : L^\infty(\Omega) &\to L^\infty(\Omega) \\ u_0 &\to S(t)u_0 = u(t,.). \end{aligned}$$

Using the techniques of R. Temam (Temam, 1997), we prove existence of attractors.

Theorem 7 *The semigroup $S(t)$, associated with the problem (1), possesses a maximal attractor A which is bounded in $W_0^{1,p}(\Omega)$, compact and connected in $L^\infty(\Omega)$.*

Proof. Inequality (6) implies that there exists an absorbing set in $L^k(\Omega), 1 \leq k \leq \infty$. We now prove the existence of an absorbing set in $W_0^{1,p}(\Omega)$ and the uniform compactness of the semigroup $S(t)$. For this purpose, multiplying (3) by $g'_{jm}(t)$, summing up from $j = 1$ to m, integrating over Ω and using Holder inequality, one obtains that

$$\left\| \frac{\partial u_m}{\partial t} \right\|_2^2 + \frac{1}{p} \frac{\partial}{\partial t} \| u_m \|^p_{W_0^{1,p}(\Omega)} \leq c_{15} \int f(u_m) \frac{\partial u_m}{\partial t} \leq c_{16}(\tau) + \frac{1}{2} \left\| \frac{\partial u_m}{\partial t} \right\|_2^2 .$$

We deduce that for all $t \geq \tau$

$$\left\| \frac{\partial u_m}{\partial t} \right\|_2^2 + \frac{\partial}{\partial t} \| u_m \|^p_{W_0^{1,p}(\Omega)} \leq c_{17}(\tau) .$$

Hence,

$$\frac{\partial}{\partial t} \| u_m \|^p_{W_0^{1,p}(\Omega)} \leq c_{17}(\tau) , \quad \forall t \geq \tau . \tag{11}$$

Multiplying (3) by $g_{jm}(t)$ we also have

$$\begin{aligned} \tfrac{1}{2} \tfrac{\partial}{\partial t} \|u_m\|_2^2 + \|u_m\|^p_{W_0^{1,p}(\Omega)} &\leq c_{18} \textstyle\int |f(u_m) u_m| \\ &\leq c_{19}(\tau). \end{aligned} \tag{12}$$

After integrating in t, we infer from the last equation (12) that

$$\int_t^{t+\tau} \|u_m\|^p_{W_0^{1,p}(\Omega)} \leq c_{20}(\tau) \quad \forall t \geq \tau. \tag{13}$$

Using (11)-(13), we can apply the uniform Gronwall Lemma (Temam, 1997, p. 89), and by the lower semi-continuity of the norm, we conclude that

$$\|u_m\|^p_{W_0^{1,p}(\Omega)} \leq c_{21}(\tau) , \quad \forall t \geq \tau .$$

It follows that the ball $B(0, c_{21}(\tau))$ of $W_0^{1,p}(\Omega)$, centered at 0 and with radius $c_{21}(\tau)$, is absorbing in $W_0^{1,p}(\Omega)$. The assumption of Theorem I.1.1 in (Temam, 1997, p. 23) is satisfied, and the proof of Theorem 7 is complete. ∎

5 Conclusions and future work

In this paper we prove existence and uniqueness for a p-Laplacian nonlinear system of partial differential equations of parabolic type, $p \geq 2$. For $p = 2$ the problem is a model of the heat diffusion produced by the Joule effect in an electric conductor, and we recover the previously known existence, boundedness, and uniqueness results found in the literature for the thermistor problem.

In a forthcoming work we will investigate the possibility to prove more regularity results of the solution of the problem, by imposing more restrictive assumptions on the data. The question is nontrivial due to the nonlinear nature of the problem, as shown in (Xu, 2004) for $p = 2$.

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